Legendrian knot theory is a rich refinement of smooth knot theory with deep connections to low-dimensional topology, symplectic and contact geometry, and singularity theory. Although smooth knot theory is a mature field with many invariants and well-explored connections between them, Legendrian knot theory has been an active area of research for only 25 years and many questions remain unanswered. My research attempts to improve upon a certain Legendrian knot invariant, understand the connections between this improved invariant and other invariants, and extend this work to Legendrian surfaces. In Part 1, I provide background material in contact topology related to Legendrian submanifolds and generating families. In Part 2, I detail my current research and propose future research projects involving Legendrian invariants derived from generating families.

1 Background

Legendrian Submanifolds and Invariants.

In $\mathbb{R}^3$, the *standard contact structure* ξ_{std} is the plane field with basis $\{\partial_y, \partial_x + y\partial_z\}$ at every point $(x, y, z) \in \mathbb{R}^3$, see Figure 1 (a). A *Legendrian knot* K in $(\mathbb{R}^3, \xi_{\text{std}})$ is a smoothly embedded submanifold diffeomorphic to S^1 with $T_pK \subset \xi_{\text{std}}\ \forall\ p \in K$, see Figure 1 (b).

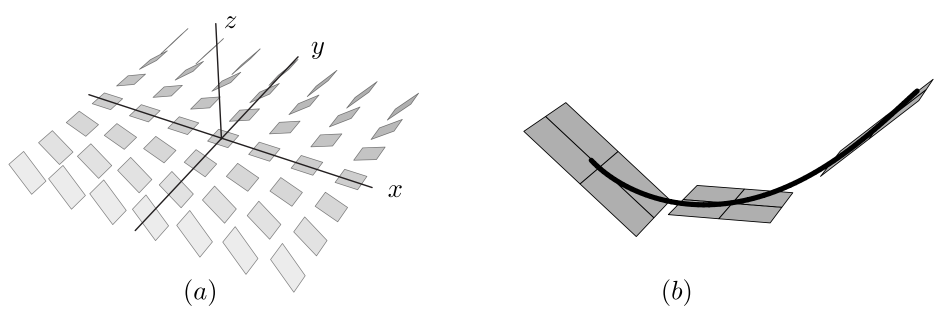

Figure 1: (a) The standard contact structure on $\mathbb{R}^3$, (b) A piece of a Legendrian knot.

We may think of ξ_{std} as the kernel of the 1-form $\alpha = dz - ydx$. In general, the standard contact structure on $\mathbb{R}^{2n+1}$ is defined as the kernel ξ_{std} of the 1-form $\alpha = dz - \Sigma_{i=1}^{n} y_i dx_i$. Heuristically, the hyperplane distribution ξ_{std} on $\mathbb{R}^{2n+1}$ is very twisty and there are no i-dimensional submanifolds $(n + 1 \le i \le 2n)$ in $\mathbb{R}^{2n+1}$ whose tangent space sits inside ξ_{std}. However, there are plenty of n-dimensional submanifolds with tangent spaces sitting inside ξ_{std}. Such submanifolds are called *Legendrian* and two Legendrian submanifolds N_0 and N_1 are said to be *Legendrian isotopic* if there exists a smooth isotopy through Legendrian submanifolds beginning at N_0 and ending at N_1.

The "classical" Legendrian knot invariants are the Thurston-Bennequin number, denoted $tb(K)$, and the rotation number, denoted $r(K)$. The first computes the twisting of ξ_{std} along K with respect to a Seifert framing of K and the second computes the rotation of the tangent space of K with respect to the standard basis on ξ_{std} and is denoted $r(K)$. Bennequin's paper [3] in 1983 provided the first connection between these geometric invariants and the genus of a Seifert surface of the knot, a purely topological invariant. His work led to new and active interest in the following questions:

Main Question. How do we determine if two Legendrian knots with identical tb and r are Legendrian isotopic? In the general case, how do we determine if two Legendrian submanifolds are Legendrian isotopic?

This first question has proved to be especially difficult to answer. Progress in the last two decades has come from using powerful geometric and topological tools, including; Floer theory and holomorphic curves [8, 5], finite dimensional Morse theory [6, 13], Khovanov homology [16], contact geometry [10, 9], and Heegaard knot Floer homology [15].

My research tackles the Legendrian knot case using generating families and Morse-Smale theory. The arguments involved intertwine geometric, algebraic and combinatorial techniques. Much of this work extends to closed, orientable Legendrian surfaces in $\mathbb{R}^5$. In Section 2, I will outline an improvement on the graded normal ruling invariant and indicate how this new Legendrian knot invariant may extend to an invariant of Legendrian surfaces.

It is possible to project away some of the coordinates in $\mathbb{R}^{2n+1}$ without losing the contact structure information. In particular, this allows us to "see" the Legendrian surfaces in $\mathbb{R}^5$ as surfaces in $\mathbb{R}^3$. The *front projection* of $\mathbb{R}^{2n+1} = \{(x_1, \dots, x_n, y_1, \dots, y_n, z)\}$ projects away the $y_1, \dots, y_n$ coordinates. Thus the front projection of a Legendrian knot lives in the (x, z)-plane of $\mathbb{R}^3$ and the front projection of a Legendrian surface lives in the (x_1, x_2, z) subspace of $\mathbb{R}^5$, see Figure 2. We denote the front projections of K and S by Σ_K and Σ_S respectively.

Σ_K and Σ_S will have self-intersections and singularities. The contact condition $T_pK \in \xi_{\text{std}} = \ker(dz - ydx)$ implies: Σ_K never has a vertical tangent space, hence we see cusps in Σ_K; the y-coordinate of $p \in \Sigma_K$ can be recovered by $y = \frac{dz}{dx}$; and at self-intersection points, the strand with the smaller slope always lies above the other strand.

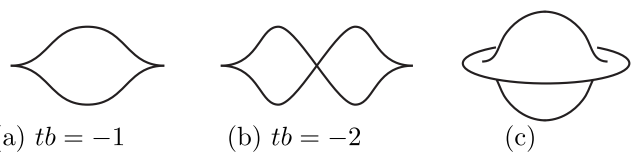

Figure 2: Front projections of two Legendrian unknots in $\mathbb{R}^3$ and a Legendrian sphere in $\mathbb{R}^5$.

The front projection allows us to calculate many Legendrian invariants combinatorially. For example, $tb(K) = \text{writhe}(\Sigma_K) - \frac{1}{2}(\#\text{cusps})$. Using this equation, we can calculate the tb of the two unknots in Figure 2 and, noting that they are different, conclude these unknots are not Legendrian isotopic. There also exists a Reidemeister-type theorem for Legendrian isotopy of Legendrian knots. Two Legendrian knots K_0 and K_1 are Legendrian isotopic if and only if their front diagrams are related by a finite sequence of the moves pictured in Figure 3. These moves facilitate a combinatorial approach to Legendrian knot theory.

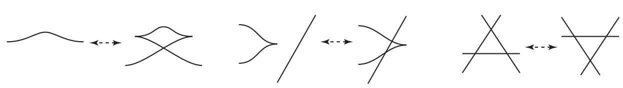

Figure 3: Legendrian Reidemeister moves for front projections.

Generating Families

It is sometimes possible to completely encode a Legendrian submanifold using a function on $\mathbb{R}^N$. Such a function is called a *generating family* and, for Legendrian knots, it leads to a combinatorial invariant derived using Morse-Smale theory and algebraic topology.

Definition 1.1. Suppose $F : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}$, $(\mathbf{x}, \mathbf{v}) \mapsto F(\mathbf{x}, \mathbf{v})$ is a smooth function, thought of as an n-parameter family of functions $F(\mathbf{x}, \cdot) : \mathbb{R}^k \to \mathbb{R}$ where $\mathbf{x} \in \mathbb{R}^n$. For a fixed $\mathbf{x} \in \mathbb{R}^n$, we let $F_{\mathbf{x}}$ denote the function $F(\mathbf{x}, \cdot)$ and refer to $\mathbb{R}^n$ as the *base space* of F. If F satisfies a certain transversality condition, then

$$K_F = \{(\mathbf{x}, \partial_{\mathbf{x}} F(\mathbf{x}, \mathbf{v}), F(\mathbf{x}, \mathbf{v})) \mid \partial_{v_i} F = 0 \text{ for } i = 1, 2, ..., k\} \subset \mathbb{R}^{2n+1} \tag{1.1}$$

is a smooth (possibly immersed) Legendrian submanifold. We say F *generates* K_F or F is a *generating family* for K_F. For $n = 1$, the front projection of K_F is simply the graph of the critical values of $F_{\mathbf{x}}$ as $\mathbf{x}$ varies, which is also known as the Cerf diagram of F.

The existence of a generating family for a Legendrian submanifold is a Legendrian isotopy invariant [23, 22, 14]. For example, Figure 2 (a) has an associated generating family, but Figure 2 (b) does not, hence we see again that these unknots are not Legendrian isotopic. The generating family for Figure 2 (a) can be seen in Figure 4 where the 1-parameter family of functions from $\mathbb{R}$ to $\mathbb{R}$ comprising the generating family sits below the unknot.

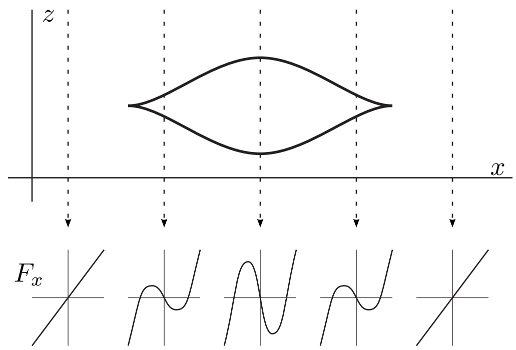

Figure 4: A generating family for the standard Legendrian unknot.

We think of a generating family as an n-parameter family of functions from $\mathbb{R}^k$ to $\mathbb{R}$. Generically, $F_{\mathbf{x}}$ will be a Morse function, but as $\mathbf{x}$ varies, $F_{\mathbf{x}}$ may have a degenerate critical point for certain $\mathbf{x}$. A point on a generic front projection will have a generating family locally even though globally this may not be true. Understanding the possible local front projections is equivalent to understanding the possible degenerate critical points that may arise in an n-parameter family of functions. For small n, the classification of such singularities was completed in [1]. Hence we can classify the local neighborhoods of front projections of Legendrian knots and surfaces. In fact, combining work of Arnol'd and Zakalyukin [1, 24] with my own combinatorial arguments, it is possible to construct a complete Legendrian Reidemeister theory for Legendrian surfaces. Using Mathematica we can visualize the Reidemeister moves that occur in a Legendrian surface. Though the Legendrian Reidemeister theory for surfaces involves considerably more cases than for knots, it is still a manageable theory that may be useful in understanding new Legendrian surface invariants.

2 Current and Future Research

Current Research

From a generating family F for a Legendrian knot K, it is possible to define a combinatorial object called a *graded normal ruling* that encodes information about a family of Morse homology complexes coming from F. For the sake of simplicity, we abbreviate these objects as *gn-rulings*. The set of all gn-rulings for a particular Legendrian knot K is a Legendrian knot invariant. This invariant has been particularly powerful in distinguishing Legendrian knots with identical tb and r [6, 13] and is intimately related to coefficients in the Kauffman and HOMFLY polynomials [20]. My current research aims to improve this invariant and extend this improved invariant to Legendrian surfaces.

Constructing gn-rulings involves understanding the evolution of a 1-parameter family of Morse-Smale complexes. Generically and regardless of dimension, the function $F_{\mathbf{x}} : \mathbb{R}^k \to \mathbb{R}$ is Morse-

Smale, thus we can form a chain complex $\mathcal{C}_{\mathbf{x}}$ with trivial homology. The critical points of $F_{\mathbf{x}}$ are the generators of $\mathcal{C}_{\mathbf{x}}$ and can be ordered by their indices and critical values. It is a classical result in algebraic topology that in such an ordered chain complex there exists a canonical pairing of the generators of $\mathcal{C}_{\mathbf{x}}$ [2]. The way this pairing changes as $\mathbf{x}$ moves through a generic 1-parameter family is determined by the possible changes in $\mathcal{C}_{\mathbf{x}}$ and is detailed in [2].

Keeping track of the pairing as $\mathbf{x}$ varies gives us the graded normal ruling, which on the front projection of the knot looks like a pairing of strands between cusps. Figure 5 gives the three gn-rulings associated to a particular Legendrian trefoil. In all three pictures, the two darker strands are paired and the two lighter strands are paired.

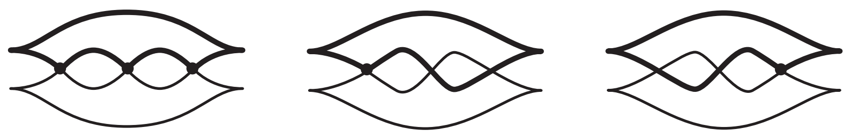

Figure 5: The three gn-rulings for the standard Legendrian trefoil.

Given that the canonical pairing on the chain complex $\mathcal{C}_{\mathbf{x}}$ exists regardless of the dimension of the base and depends only on the function $F_{\mathbf{x}}$ and not on any global condition, it is reasonable to expect that there exists a similar combinatorial invariant for the front projection of a Legendrian surface. Finding such an invariant is one focus of my current research.

The first attempt at such a Legendrian surface invariant involves slicing the surface with planes and restricting to the Legendrian knot case. Given a Legendrian surface S with generating family F, there exist suitably generic curves γ in the base space $\mathbb{R}^2$ of F with the property that $F|_\gamma$ generates a Legendrian knot K_γ with front projection Σ_γ and a gn-ruling on Σ_γ. We call such curves *slicing curves*. A reasonable definition of a gn-ruling on a Legendrian surface S would be a combinatorial object T that satisfies:

Condition 2.1. *The restriction of T to any slicing curve γ gives a gn-ruling on Σ_γ.*

The appropriate combinatorial object that encodes Condition 2.1 is a labeling of the lines of self-intersection of the front projection Σ_S by marks indicating whether or not the pairing switches when a curve γ crosses a line of self-intersection. We will call such a labeling a *ruling label*. This may seem like a potentially difficult object to work with, especially since the set of all slicing curves is dense in $\mathbb{R}^2$. However, we have the following theorem:

Theorem 2.2 (MBH)**.** *There exists a basis of slicing curves $\gamma_1, \ldots, \gamma_n$ and a finite-time algorithm that computes the set of all ruling labels of Σ_S from the gn-rulings of $\Sigma_{\gamma_1}, \ldots, \Sigma_{\gamma_n}$.*

However, ruling labels do not encode enough of the generating family information to be a Legendrian surface invariant. The problem arises in the definition of a gn-ruling. A gn-ruling encodes the pairing of critical points in the family of chain complexes $\mathcal{C}_{\mathbf{x}}$, but it neglects to take into account the possible geometric handleslides that occur in a 1-parameter family of functions $F_{\mathbf{x}}$. These handleslides change $\mathcal{C}_{\mathbf{x}}$, but do not change the pairing of critical points, and hence they are "forgotten" by the gn-ruling invariant. In dimension 1, this is not a problem, but these handleslides are essential in the development of the two dimensional invariant. In fact, it now seems unlikely that a purely combinatorial invariant can be defined on Legendrian surfaces by restricting to the 1-dimensional gn-ruling invariant. On the other hand, refining the definition of a gn-ruling to include handleslide information may allow us to define a combinatorial invariant on Legendrian surfaces using the slicing curves.

The problem of incorporating handleslides into the gn-ruling theory was taken up in a small group session at the American Institute of Mathematics (AIM) workshop on Legendrian and Transverse Knots in September 2008. Over five days, our group of eight[1] sought to find the appropriate way of adding handleslide data to the gn-rulings. We were motivated by the recent work of Fuchs and Rutherford in [13] and ideas suggested by P. Pushkar in a personal correspondence with D. Fuchs in 2001 [18]. We made substantial progress towards understanding the appropriate "decorations" that need to be added to the gn-rulings and formulated the definition of a *decorated normal ruling*, abbreviated *decorated gn-ruling*. In fact, our small group has applied for a SQuaREs grant from AIM that will allow us to reconvene at AIM for up to two weeks to continue our collaboration on this project.

On the concluding day of the AIM workshop, D. Fuchs received an email from P. Pushkar outlining progress he has made on this problem. In [19], Pushkar proposes decorating the gn-rulings with vertical markers to indicate where handleslides occur in the 1-parameter family of chain complexes. Pushkar refers to these vertical markers as *springs* and uses them to form *spring complexes*. These spring complexes are equivalent to our group's decorated gn-rulings. He claims, under an appropriate equivalence relation, there is a natural bijection between the sets of decorated gn-rulings of two Legendrian isotopic front projections. Pushkar also outlines an extension of decorated gn-rulings to an invariant of Legendrian surfaces. Many of the techniques in his extension correspond with techniques developed in the proof of Theorem 2.2, though his theory is slightly more developed than my own. Pushkar has not yet published his work on spring complexes.

I am currently working to prove that, under the appropriate equivalence relation, decorated gn-rulings do indeed form a Legendrian knot invariant. The equivalence relation is inspired by the work of Pushkar in [19]. The equivalence relation and invariance arguments require understanding the possible interactions between geometric handleslides, but once these interactions are understood, the proof techniques are largely combinatorial and algebraic in nature. This work is very nearly complete. As an example, Figure 6 shows the five possible decorated gn-rulings of a Legendrian trefoil. I have also made significant progress towards modifying the definition of ruling labels so as to extend the decorated gn-ruling invariant from Legendrian knots to an invariant of Legendrian surfaces. The new surface invariant will respect Condition 2.1 and evidence suggests the algorithm and basis used in Theorem 2.2 will allow us to compute the new invariant on a Legendrian surface.

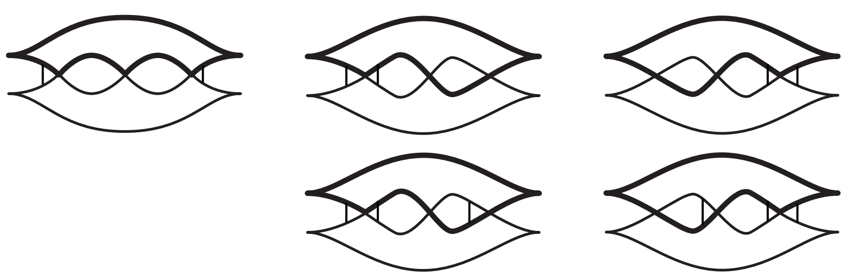

Figure 6: The five decorated gn-rulings for the standard Legendrian trefoil.

Future Research

The decorated gn-ruling is meant to encode the geometric handleslides that exist in a generating family. However, decorated gn-rulings are still combinatorial objects and their exact relationship

[1]Group Members: Sergei Chmutov (Ohio State), Dimitry Fuchs (UC Davis), Victor Goryunov (Univ of Liverpool), Paul Melvin (Bryn Mawr), Dan Rutherford (Duke), Joshua Sabloff (Haverford), Lisa Traynor (Bryn Mawr).

with generating families is not clear. Every generating family induces a decorated gn-ruling, but the converse is not known.

Question 1. Can every decorated gn-ruling of Σ_K be realized by a generating family for K?

An affirmitive answer to this question would provide a very strong link between the geometric nature of generating families and the combinatorial nature of front projections. Such a link would open avenues to studying Questions (2), (3), and (4).

In [6], Chekanov and Pushkar define an index on gn-rulings and use combinatorial arguments to show that the number of gn-rulings of a particular index is invariant under the Legendrian Reidemeister moves in Figure 3. By counting gn-rulings of a particular index, they form a polynomial invariant called the *gn-ruling polynomial*.

Question 2. Is there a way to index decorated gn-rulings to create a similar polynomial invariant? If so, do there exist Legendrian knots distinguished by this polynomial but not by gn-ruling polynomial?

Decorated gn-rulings may provide insight into several open problems in Legendrian knot theory. Most notably, decorated gn-rulings may enhance an existing link between generating families and the Chekanov-Eliashberg differential graded algebra. In 2000, Chekanov and Eliashberg independently created a differential graded algebra invariant (named the Chekanov-Eliashberg DGA and abbreviated C-E DGA) [8, 4]. The C-E DGA is a special case of Symplectic Field Theory [8, 7], which generalizes work of Floer and Gromov. The C-E DGA can be difficult to compute even for simple knots, but the theory has a combinatorial translation that allows us to apply topological and combinatorial arguments to extract computable Legendrian invariants. This theory uses the *Lagrangian projection* of a Legendrian knot onto the (x, y)-plane, instead of the front projection.

Given a Legendrian knot K with Lagrangian projection L, the C-E DGA is an algebra A_L with a graded basis $\{q_1, \dots, q_n\}$ corresponding to the crossings of L and a differential ∂ which counts certain immersed disks. An important Legendrian invariant extracted from the C-E DGA is the *graded augmentation number*, which is a count of all algebra maps $\epsilon : A_L \to \mathbb{Z}/2$ that satisfy $\epsilon \circ \partial = 0$, $\epsilon(1) = 0$, and if $\epsilon(q_i) = 1$ then the grading of q_i is 1.

Even though generating families and the C-E DGA involve very different analytic techniques, there exist remarkable, and still somewhat mysterious, connections between these two invariants.

Theorem 2.3 ([11, 12, 21, 17])**.** *The C-E DGA of a Legendrian knot K has a graded augmentation if and only if the front projectoin of K has a gn-ruling. In fact, if we fix certain front and Lagrangian projections of K, then there exists a many-to-one correspondence between graded augmentations and gn-rulings.*

Given that decorated gn-rulings are a refinement of gn-rulings, it is natural to ask:

Question 3. What is the relationship between decorated gn-rulings and augmentations? Is there a bijection between the set of decorated gn-rulings and augmentations?

Simple examples have so far indicated a bijection might exist. Pushkar also indicates in [19] that such a bijection does exist.

Given a generating family F for K it is possible to form a chain complex giving homology groups $\mathcal{G}H_*(F)$. In [13], Fuchs and Rutherford draw a correspondence between the generating family homology $\mathcal{G}H_*(F)$ and the linear contact homology $H^{\epsilon}_*(L)$ coming from an augmentation. A bijection between decorated gn-rulings and augmentations may improve this correspondence. We may also ask about the relationship between $\mathcal{G}H_*(F)$ and the decorated gn-ruling coming from F:

Question 4. Do two generating families F_0, F_1 inducing the same Legendrian knot K and identical decorated gn-rulings necessarily give isomorphic homology groups, $\mathcal{G}H_*(F_0) \cong \mathcal{G}H_*(F_1)$?

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