The beautiful complexity of a simple knotted circle in three dimensional space fascinates and frustrates mathematicians and holiday-light enthusiasts alike. In a matter of minutes a student can be brought to understand the basic objects (knots) and the basic operations (deforming them in 3-space) of knot theory; see Figure 1. The student's natural intuition provides a firm starting point and confidence to dig deeper. A few well-placed examples can stoke an ember of interest into a wildfire. Soon the student is sitting in a coffee shop playing with shoe laces and colored pencils, the recipient of sideways glances from others nearby. I have traveled this path and I have the bits of string to prove it. The knots I hold dear nowadays are a bit different, but they have the same magnetic capacity to bring students to higher mathematics. In my research, I investigate a class of smooth knots called Legendrian knots. Symplectic Field Theory and the theory of generating families are two foundational tools in Legendrian knot theory with very different origins. Roughly speaking, my research has shown, and continues to show, that deep connections exist between these two theories.

Figure 1: (a) Two knotted circles represented by their projection to a 2-dimensional plane. Can the second be deformed into the first? (b) Deforming a smooth knot in 3-space is equivalent to a sequence of these moves.

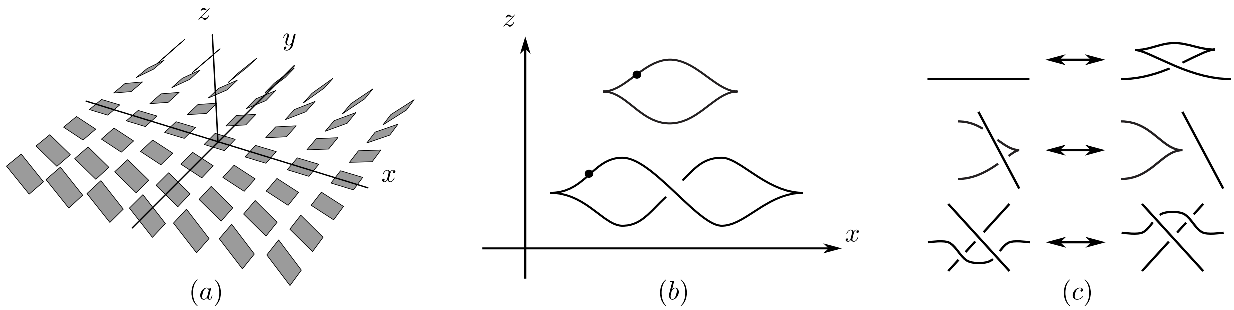


Figure 2: (a) Planes in the standard contact structure on $\mathbb{R}^3$. (b) The xz-projections of two Legendrian knots. The two identified points both have slope 1 and so their y coordinate in $\mathbb{R}^3$ is 1. (c) A deformation of Legendrian knots is equivalent to performing a sequence of these moves on an xz-projection.

Legendrian knots originate in the study of two dimensional analogues to vector fields. At each point $(x, y, z) \in \mathbb{R}^3$, we consider the 2-dimensional plane spanned by the vectors ∂_y and $\partial_x + y\partial_z$. The *standard contact structure* on $\mathbb{R}^3$ is the collection of all of these planes; see Figure 2 (a). A *Legendrian knot* is a smooth knot whose tangent space at each point sits within the plane at that point. We visualize Legendrian knots by projecting them to the xz-plane. The resulting planar curve has cusps since the contact structure planes are never completely vertical; see Figure 2 (b).

The projected y coordinate is the slope of the tangent line to the path.

Contact structures are an important geometric tool in the long-term project to understand and classify the topology and geometry of 3-dimensional manifolds. I found my way to Legendrian knots through this larger project. Important constructions and results in low-dimensional topology rely on our ability to distinguish Legendrian knots.

Two Legendrian knots are equivalent if one can be deformed into the other through Legendrian knots. Such a deformation is equivalent to changing the xz-projection by a sequence of the moves in Figure 2 (c). Our intuition for smooth knots can be misleading in this new setting. For example, the two knots in Figure 2 (b) are equivalent as smooth knots, but they are not equivalent as Legendrian knots. This is our first indication of the subtlety of Legendrian knot theory. We distinguish Legendrian knots by constructing *invariants*, mathematical objects assigned to a knot that remain unchanged as we deform the knot. Smooth knot theory has hundreds of invariants. However few Legendrian knot invariants exist, those that do are not fully understood, and many basic questions remain unanswered. In short, this field is accessible, relevant, young and open to undergraduate involvement.

In the next section, we will use the Legendrian trefoil K in Figure 3 to walk through two constructions used to create Legendrian invariants. Both constructions assign an algebraic object to K and my recent work has shown that, though they are built upon seemingly unrelated geometric foundations, they are closely related. The algebraic objects have combinatorial formulations that can be described by a set of basic rules understandable to advanced undergraduates. In Section 3, I detail a future project in which undergraduates can make important, original contributions.

1 Two approaches to Legendrian knot invariants

Our first approach assigns to K an algebra called the *Chekanov-Eliasberg differential graded algebra* [2, 1], abbreviated CE-DGA. The geometric origins of the CE-DGA are analytically intensive, depending on the Symplectic Field Theory of [2]. However, the entire enterprise has a straightforward and algorithmically computable description. The algebra $\mathcal{A}(K)$ is generated by monomials in the labels a, b, c, d, e assigned to the crossings and right cusps of K. We define a map $\partial : \mathcal{A}(K) \to \mathcal{A}(K)$ that counts disks in the xz-plane with convex corners at the crossings and cusps of K; see Figure 3.

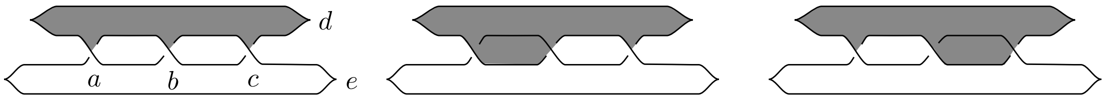
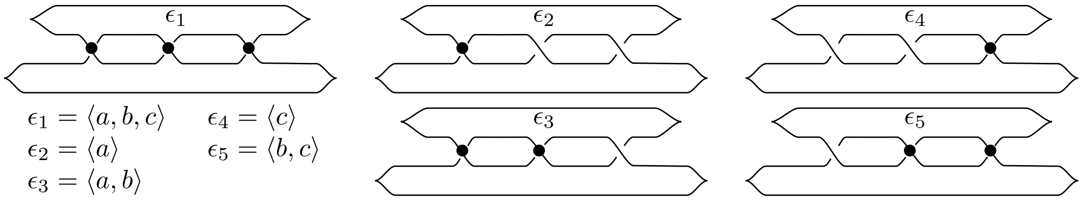


Figure 3: Convex disks on a Legendrian trefoil. Each disk contributes a monomial to $\partial(d) = 1 + abc + c + a$ and corresponds to the projected image of a pseudo-holomorphic disk in $\mathbb{R}^4$.

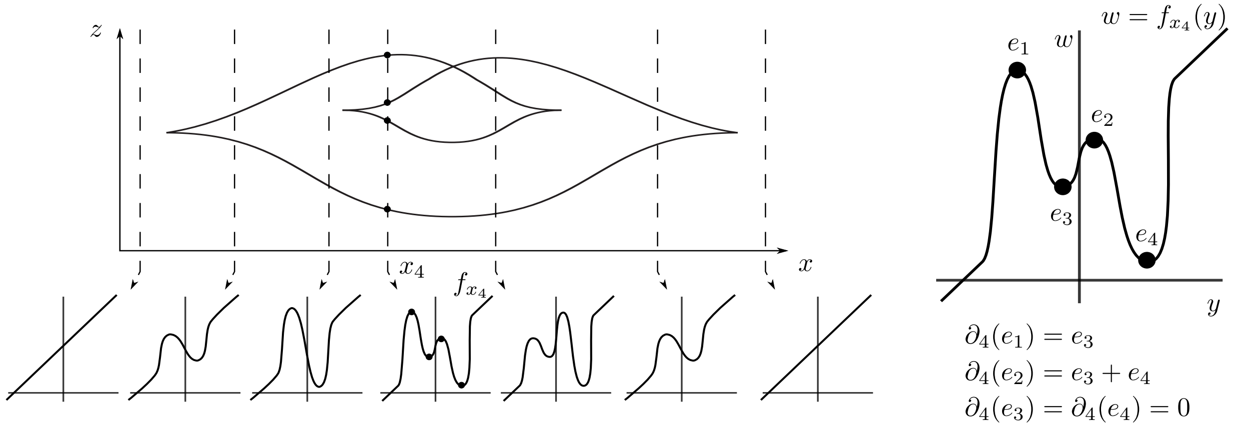


$\epsilon_1 = \langle a, b, c \rangle$ $\epsilon_4 = \langle c \rangle$
$\epsilon_2 = \langle a \rangle$ $\epsilon_5 = \langle b, c \rangle$
$\epsilon_3 = \langle a, b \rangle$

Figure 4: The five augmentations of K. The bracket notation indicates the crossings on which the augmentation takes the value 1.

As an invariant, the CE-DGA is easy to construct, but its infinite dimensional, noncommutative nature makes it difficult to use. We instead try to chip away new invariants from $(\mathcal{A}(K), \partial)$ using tools that remove some of the complexity from $\mathcal{A}(K)$. The investigation of augmentations is one such example. An *augmentation* ϵ is an assignment of 0's and 1's to the crossings and right cusps of K that extends to an algebra homomorphism $\epsilon : \mathcal{A}(K) \to \mathbb{Z}_2$ satisfying $\epsilon \circ \partial = 0$; see Figure 4. Each augmentation has an associated differential algebra $(\mathcal{A}(K), \partial^\epsilon)$ isomorphic to $(\mathcal{A}(K), \partial)$. A count of the augmentations provides an invariant, as does the first level homology of $(\mathcal{A}(K), \partial^\epsilon)$.

Our second approach to Legendrian invariants is just as useful and computable as the first. A *generating family* for the Legendrian trefoil K is a 1-parameter family of smooth functions $f_x : \mathbb{R}^n \to \mathbb{R}$, $x \in \mathbb{R}$ so that $K = \{(x, f_x(w)) | w \in \mathbb{R}^n$ is a critical point of $f_x\}$; see Figure 5 (a). In essence, all of the important information of K sits within the functions f_x and so Legendrian invariants are just waiting to be extracted from these functions.



(a) (b)

Figure 5: (a) For this Legendrian knot, we describe a generating family by giving the graphs of $f_x : \mathbb{R} \to \mathbb{R}$ for several values of x. Outside of a closed interval of $\mathbb{R}$, the functions f_x are linear and so they do not have critical points. The critical points of f_{x_4} correspond to the four points of intersection in $K \cap (\{x_4\} \times \mathbb{R})$. (b) The vector space C_4 is generated by the critical points e_1, e_2, e_3, and e_4.

For a fixed x, we construct a $\mathbb{Z}_2$ vector space C_x generated by the critical points of f_x and define a map $\partial : C_x \to C_x$ by counting gradient flowlines between critical points. Visually, this amounts to pouring hot fudge onto the graph of f_x and watching the flow paths that form between critical points; see Figure 5 (b). The pair (C_x, ∂_x) is a homology chain complex and a *Morse complex sequence*, abbreviated MCS, is a finite sequence of chain complexes encoding (C_x, ∂_x), for $x \in \mathbb{R}$. An MCS can be graphically encoded by adding vertical marks on K that indicate certain gradient flow anomalies; see Figure 6.

The Legendrian trefoil in Figure 3 admits several different generating families. Although they are not easily visualized, we can determine the finite sequences of chain complexes that might arise from these generating families. In this manner, we can "see" the generating families algebraically in terms of MCSs, even though we can't write down their equations.

2 Recent Results and Future Directions

It has been known for some time that connections exist between the CE-DGA and the theory of generating families; see any of [3, 4, 7, 10]. For example, a Legendrian knot K admits a generating family if and only if $(\mathcal{A}(K), \partial)$ admits an augmentation. The depth of these connections continues

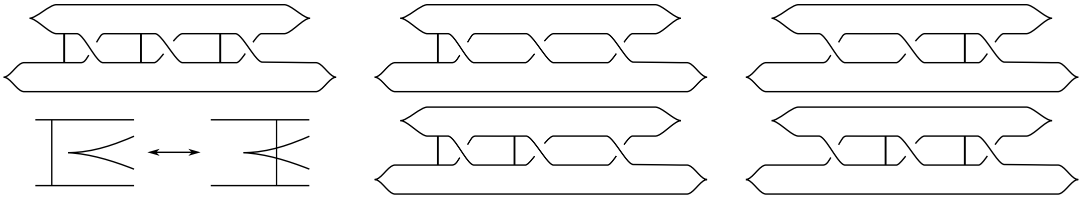


Figure 6: The five MCS classes for the Legendrian trefoil. There is a natural equivalence relation on MCSs given by a collection of local moves involving vertical marks. One such move is given in the lower left. Compare these MCSs to the augmentations in Figure 4. There's a theorem in there!

to grow. In [5], I provide the definition of an MCS[1] and prove a number of results concerning their connection with augmentations.

Theorem 2.1 (M. B. Henry [5])**.** *Suppose K is the xz-projection of a Legendrian knot. Then*

1. *K has an augmentation if and only if K has an MCS;*
2. *There is an injection from the set of augmentations of K into the set of MCSs;*
3. *If K has exactly two left cusps, then this injection can be extended to a bijection on equivalence classes; and*
4. *Every MCS is equivalent to the image of an augmentation from the injection in 2.*

The bijection in 3. is evident when you compare the augmentations in Figure 4 with the MCSs in Figure 6.

Recently, Dan Rutherford (Duke University) and I have refined the relationship between MCSs and augmentations. In [6] we condense the finite sequence of chain complexes in an MCS $\mathcal{C}$ into a single differential graded algebra $(\mathcal{A}(\mathcal{C}), \partial)$. This process corresponds to counting certain gradient flow trees in a generating family f_x. Our first results appear in [6].

Theorem 2.2 (M. B. Henry and D. Rutherford [6])**.** *Given an augmentation ϵ of K, there exists an MCS $\mathcal{C}$ so that the differential algebra $(\mathcal{A}(\mathcal{C}), \partial)$ is equal to the differential algebra $(\mathcal{A}(K), \partial^{\epsilon})$.*

We must be careful not to lose sight of the geometry as we dig deeper into the algebraic and combinatorial computations related to MCSs. One way to say this is, "If the geometry implies the algebra, does the converse relationship hold? Do MCSs and algebraic objects derived from MCSs always have geometric origins?" One current and future avenue of research investigates this question.

Conjecture 2.3. Given an MCS $\mathcal{C}$ on K, there exists a generating family $f_x : \mathbb{R}^n \to \mathbb{R}$, $x \in \mathbb{R}$ so that $\mathcal{C}$ encodes the 1-parameter family of chain complexes of f_x. In addition, the differential algebra $(\mathcal{A}(\mathcal{C}), \partial)$ is equal to the differential algebra resulting from examining gradient flow trees in the difference function $G : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ given by $G(x, v, w) = f_x(v) - f_x(w)$.

Theorems 2.1 and 2.2 and our current work towards Conjecture 2.3 moves us a bit closer to our long-term goal.

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[1] Petya Pushkar outlines a program involving Morse complex sequences in [9, 8], but his work is unpublished. His notes have had a positive impact on my own work.

Conjecture 2.4. Symplectic Field Theory and the theory of generating families give rise to identical Legendrian knot invariants. Every object derived from one theory has a corresponding object in the other and the translation between the two theories can be given explicitly.

My collaboration with Dan Rutherford continues to bear fruit and motivate new questions. The American Institute of Mathematics (AIM) workshop on Legendrian and Transverse Knots in September 2008 also inspired new collaborations. Over five days, our small group of eight[2] sought to refine existing Legendrian invariants from generating families. Since then, we have received a SQuaREs grant from AIM to reconvene there for two weeks to continue our collaboration. Our first week was in June 2009 and we are scheduled to meet again next May.

As my research moves forward, I am eager to involve undergraduates in the creation of original mathematics. Extending MCSs to Legendrian knots in the open solid torus $S^1 \times \mathbb{R}^2$ is a project perfectly suited for undergraduate research. The standard contact structure on $\mathbb{R}^3$ has a natural analogue in $S^1 \times \mathbb{R}^2$. The xz-projection of a Legendrian knot in $S^1 \times \mathbb{R}^2$ has the same cusps as xz-projections in $\mathbb{R}^3$, however it now sits on an annulus rather than on a plane. The definition of an MCS in $S^1 \times \mathbb{R}^2$ can be understood combinatorially in terms of vertical marks on the xz-projection or in terms of matrices and matrix transformations. When placed in an appropriate context, an advanced undergraduate comfortable with multivariable calculus, linear algebra, and group theory has the tools necessary to assist in the translation of MCS results from $\mathbb{R}^3$ to $S^1 \times \mathbb{R}^2$.

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