Matrix Algebra, Math 309, Summer 2007
Take-home Exam 1

July 27, 2007

On my honor, I have not given or received outside help on this exam beyond those resources outlined in the syllabus.
Sign Here: ____

1. Essay

Write an essay detailing how our investigation of linear functions $f : \mathbb{R}^n \to \mathbb{R}^m$ (i.e. "nice" functions from lists of numbers to lists of numbers) led to the development of basis vectors, matrices, matrix-vector multiplication, and matrix-matrix multiplication. You should include how each of these objects or operations were motivated by linear functions. You should also include examples to illustrate your points.

A thorough essay should take 300-400 words, which is a little more than one page double-spaced in Times New Roman 12pt font. You do not need to type your essay, but it should be well organized and in essay form.

2. Let

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 4 & 1 \\ 4 & 1 & 2 \end{bmatrix}$$

(a) Find the matrices of the LU factorization of A.

(b) Using the LU factorization of A, show that there is a row-reduced system of linear equations that is equivalent to $A\mathbf{x} = \mathbf{b}$. Use this already row-reduced system to solve$A\mathbf{x} = \mathbf{b}$ for

$$\mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Recall that a "row-reduced system of linear equations" is a system of the form $B\mathbf{x} = \mathbf{c}$ where B is a row-reduced matrix.

3. Let A be an invertible $n \times n$ matrix.
(a) Show that $\operatorname{rref}(A) = I_{n \times n}$.
(b) Explain how to find the i^{th} column of A^{-1} without finding all of A^{-1}.

4. A linear transformation f from $\mathbb{R}^n$ to $\mathbb{R}$ is called a *linear functional.* These types of maps come in very handy in a variety of academic areas. Here are a few exercises to help you understand how they work.

(a) Let a be a vector in $\mathbb{R}^n$. Define a function $f : \mathbb{R}^n \to \mathbb{R}$ by $f(\mathbf{x}) = a \cdot \mathbf{x}$, where $\cdot$ indicates the dot product. Show that f is a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}$, i.e. show f is a linear functional.

It turns out that every linear functional is of the type found in (a), i.e. they all are just the dot product of the input vector $\mathbf{x}$ with some fixed constant vector a.

(b) Let $g : \mathbb{R}^n \to \mathbb{R}$ be any linear functional and let

$$\mathbf{a}_g = \begin{bmatrix} g(\mathbf{e}_1) \\ g(\mathbf{e}_2) \\ \vdots \\ g(\mathbf{e}_n) \end{bmatrix}$$

where $\mathbf{e}_1, \dots, \mathbf{e}_n$ are the standard basis vectors in $\mathbb{R}^n$.

Show that $g(\mathbf{x}) = A_g\mathbf{x} = \mathbf{a}_g \cdot \mathbf{x}$ for all $\mathbf{x}$ in $\mathbb{R}^n$. If you do not recall what A_g is, look in Chapter 2, section 2.

(c) Let $h : \mathbb{R}^n \to \mathbb{R}$ be a linear functional and let $\mathbf{a}$ and $\mathbf{b}$ be vectors in $\mathbb{R}^n$. Suppose for all $\mathbf{x}$ in $\mathbb{R}^n$, $h(\mathbf{x}) = \mathbf{a} \cdot \mathbf{x}$ and $h(\mathbf{x}) = \mathbf{b} \cdot \mathbf{x}$. Show that $\mathbf{a} = \mathbf{b}$.

This tells us that the vector a_g in part (b) is the *unique* vector in $\mathbb{R}^n$ for which $g(\mathbf{x}) = \mathbf{a}_g \cdot \mathbf{x}$ for all $\mathbf{x}$ in $\mathbb{R}^n$.

(d) Define a function $f : \mathbb{R}^3 \to \mathbb{R}$ by

$$f\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = a + 2b + 4c$$

Show that f is a linear functional and find the vector $\mathbf{a}_f$ in $\mathbb{R}^3$ so that $f(\mathbf{x}) = \mathbf{a}_f \cdot \mathbf{x}$.

5. Let l be a line in $\mathbb{R}^3$ and $\mathbf{y}$ be any point in $\mathbb{R}^3$. This exercise will find which point on l is closest to $\mathbf{y}$. We will call the distance from $\mathbf{y}$ to this closes point the *distance from* $\mathbf{y}$ *to* l.

Suppose that $\mathbf{x}(t) = \mathbf{x}_0 + t\mathbf{v}$ is a parameterization of l. The shortest distance between l and $\mathbf{y}$ is the value of t that makes $|\mathbf{x}(t) - \mathbf{y}|^2$ as small as possible.

(a) Show that $|\mathbf{x}(t) - \mathbf{y}|^2 = |\mathbf{v}|^2 t^2 - 2t\mathbf{v} \cdot (\mathbf{y} - \mathbf{x}_0) + |\mathbf{y} - \mathbf{x}_0|^2$

(b) Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by

$$f(t) = |\mathbf{v}|^2 t^2 - 2t\mathbf{v} \cdot (\mathbf{y} - \mathbf{x}_0) + |\mathbf{y} - \mathbf{x}_0|^2$$

Using calculus or the completing the square technique, show that $t = \frac{\mathbf{v}\cdot(\mathbf{y}-\mathbf{x}_0)}{|\mathbf{v}|^2}$ minimizes $f(t)$. If you get stuck on this, just assume $t = \frac{\mathbf{v}\cdot(\mathbf{y}-\mathbf{x}_0)}{|\mathbf{v}|^2}$ does minimize $|\mathbf{v}|^2 t^2 - 2t\mathbf{v} \cdot (\mathbf{y} - \mathbf{x}_0) + |\mathbf{y} - \mathbf{x}_0|^2$ and move on to part (c).

(c) Using the value of t given by part (b), deduce that the point on l that is closest to $\mathbf{y}$ is the point

$$\mathbf{x}_0 + \left(\frac{\mathbf{v} \cdot (\mathbf{y} - \mathbf{x}_0)}{|\mathbf{v}|^2} \right) \mathbf{v}$$

Let $\mathbf{z} = \mathbf{x}_0 + \left(\frac{\mathbf{v}\cdot(\mathbf{y}-\mathbf{x}_0)}{|\mathbf{v}|^2} \right) \mathbf{v}$.

(d) Show that

$$|\mathbf{z} - \mathbf{y}|^2 = |\mathbf{y} - \mathbf{x}_0|^2 - \frac{|\mathbf{v} \cdot (\mathbf{y} - \mathbf{x}_0)|^2}{|\mathbf{v}|^2}$$

and then show

$$|\mathbf{z} - \mathbf{y}|^2 = |(\mathbf{y} - \mathbf{x}_0)_\perp|^2$$

where

$$\mathbf{y} - \mathbf{x}_0 = (\mathbf{y} - \mathbf{x}_0)_\| + (\mathbf{y} - \mathbf{x}_0)_\perp$$

is the decomposition of $\mathbf{y} - \mathbf{x}_0$ into its components that are parallel and perpendicular to $\mathbf{v}$.

So the distance from $\mathbf{y}$ to l is $|(\mathbf{y} - \mathbf{x}_0)_\perp|$.

Extra Credit: Draw an example picture in $\mathbb{R}^3$ that includes $l, \mathbf{y}, \mathbf{x}_0, \mathbf{y}-\mathbf{x}_0, (\mathbf{y}-\mathbf{x}_0)_\|, (\mathbf{y}-\mathbf{x}_0)_\perp$, and $\mathbf{z}$.

Explain why it makes sense geometrically that $\mathbf{x}_0 + \left(\frac{\mathbf{v}\cdot(\mathbf{y}-\mathbf{x}_0)}{|\mathbf{v}|^2} \right) \mathbf{v}$ is the closest point on l to $\mathbf{y}$ and that the distance from $\mathbf{y}$ to l is $|(\mathbf{y} - \mathbf{x}_0)_\perp|$.